Model Theory in Logic

Early in the development of modern logic it was shown that any theory expressed in English could be completely translated into arithmetic (and back again). This meant that mathematical methods could be used to study the consistency and structure of any well expressed theory. The result is a specialist branch of analytic philosophy, in which theories of the sciences and of metaphysics can be studied with great rigour, with new insights emerging from the formal methods. Whether this is now the cutting edge of analytic philosophy or a very interesting detour remains to be decided, but all analytic philosophers should be aware of some of the principles and techniques involved. The main technique of this approach is Model Theory, which studies logical structures by means of set theory.

Model theory focuses on first-order logic (with a domain just containing objects), and studies finite domains. Eventually the subject encounters second-order quantifying (over predicates, relations and subsets), and has to confront the large mathematical claims about infinity, but at this point model theory falters, and intense debates begin. A **theory** is set of sentences expressed in a precise logical language, and is subject to the syntactic rules of that language. It starts from an initial set of sentences, the axioms, and the theory is **closed** under the implications of the rules, meaning it contains all possible deductions from them. So far the theory is just syntactic, but we then add an **interpretation**, meaning a semantic assignment of either T or F to every sentence. A large theory has a vast number of interpretations (involving all the permutations of T or F for the sentences), and each interpretation is a **model**. A model is semantically **consistent** if there is at least one model in which all the sentences can be assigned T.

The assignment of Ts and Fs to sentences could be done directly, but it is usually **compositional**, meaning that the truth-value of a sentence is built up from its components. The options for T and F are restricted by the domain of objects (a set), which could be quite small, and the predicates, which are subsets of the objects. A compositional assignment builds atomic sentences by assigning predicates to objects, and the atomic sentences can then be combined to a higher level of complexity. Thus the theory has a **structure**, which is the main focus of model theory. A distinguishing feature of each model is its **cardinality**, which is the number of objects in its domain. Mathematical models introduce functions and relations, which are also expressed using set theory.

We can assess whether an individual model is tautological, consistent or impossible, but much of model theory deals with comparisons and relations between different models, looking for important resemblances and differences. A **function** is a set of ordered pairs, consisting on an input and an output. By the systematic application of a function to all of its elements, a domain can be converted into a new model, or can replicate an existing model. For any function we have a source set A (the 'domain') and an output set B (the 'range'), written as f:A→B. We are interested in four main options for how this activity proceeds.

A function is '**general**' if each element of A produces some output in B. A-elements can only have a single output, but B-elements could be produced twice, from different sources in A. A function is '**surjective**' (said to be '**onto**' its range) if every B-element results from some A-element, so that every B-element is accounted for by the function. If the function is non-surjective then B will contain elements that were not created by the function. A function is '**injective**' ('**one-to-one**') if each output element in B is accounted for by just one element of A, so that each application of the function has a unique output. If the function is injective but not surjective, there will be B-elements that are not accounted for by the function. A function is '**bijective**' (both 'onto' and 'one-to-one') if every B-element is accounted for by a unique A-element in. The most significant relation between models that emerges from this is that when a function can fully replicate a set, with nothing missing, then the two models are '**isomorphic**', which in mathematics means they are indistinguishable for all practical purposes. If all of the models of a theory are isomorphic to one another, then the whole theory is '**categorical**', which is the highest level of success in modelling (achieved by the second-order Peano Axioms of arithmetic, for example).

Three major theorems have emerged from the study of models, by which they can be compared, and their strengths and weaknesses revealed. Semantic consistency of a theory is proved by a single model with all sentences true, and syntactic consistency is proved if every propositions or its negation is provable, and never both. The ideal of a system of logic (achieved buy first-order logic) is for these to coincide, when everything provable can be modelled and everything modellable can be proved, making the system **complete**. A first-order logical system is said to be **compact** when any provable sentence can be proved using finite means, even if the theory extends into infinity. It can be proved that consistent first-order systems are compact. This is good and bad news, because it brings infinite proofs within our grasp, but blocks expansions of the logic into the infinite. Compact logics are good for ordinary reasoning, but not so good for mathematics. The third theorem is the **Löwenheim-Skolem Theorem** (or property), which concerns movement between smaller and larger models. The **upward** version of the theory says that if a theory has an infinite model, then the model can always be expanded to a model of higher cardinality. The **downward** version says that (if the language is 'countable') a consistent theory will always have a countable model. Between them they show the limitations of first-order logic, which cannot discriminate between different infinite cardinalities. A minority view thus spurns large infinities, but the majority become sympathetic to second-order logic.

A striking use of finite model theory (which is a successful enterprise, with none of the problems of large infinities) is to raise doubts about our ability to talk accurately about reality. A model 'satisfies' a theory if there is an assignment of a set of objects which makes all of its sentences true. The problem is that there is never just one successful model, as there have to be many isomorphic models with the same domain of objects. This seems to imply that even if our talk about reality works, it could have worked just as well if we had systematically rearranged what it refers to. Hence there could never be the 'one true theory' of reality. It is arguments of this kind which attract philosophers to model theory. Mastery of the area takes intensive study, but it seems to reveal realms of logical truth that have a life of their own, and offer a tantalising glimpse into a different sort of rational reality.